

RG Invariance of the Pole Mass in the Minimal Subtraction Scheme

Chungku Kim

(Dated: August 30, 2012)

We prove the renormalization group(RG) invariance of the pole mass with respect to the RG functions of the minimal subtraction(MS) scheme and illustrate this in case of the neutral scalar field theory both in the symmetric and in the broken symmetry phase.

PACS numbers: 11.15.Bt, 12.38.Bx

I. INTRODUCTION

The pole mass defined as the pole of the propagator plays an important role in the process where the characteristic scale is close to the mass shell[1]. The one-loop relation between the scalar quartic coupling constant and the pole mass of the standard model was obtained long ago[2] and the two-loop pole mass of the standard model has been obtained later[3]. Recently, it was shown that the pole mass is infrared finite and gauge invariant[4]. The RG evolution of the pole mass in the perturbative calculation was used to determine the bound of the Higgs mass[5]. In this paper, we will prove that the pole mass when expressed in terms of the Lagrangian parameters, satisfies the RG equation with respect to the RG functions of the MS scheme both in the symmetric phase and in broken symmetry phase. In case of the broken symmetry phase ($m^2 < 0$), we will follow an approach where we first determine the VEV as a function of the Lagrangian parameters from the effective potential(EP) in the symmetric phase ($m^2 > 0$) renormalized in the MS scheme. Then by shifting the field to one with vanishing VEV, we obtain the Lagrangian in broken symmetry phase which contains both the usual $\frac{1}{\epsilon}$ divergent counter-terms and the finite counter-terms coming from the $O(\hbar)$ terms of the VEV's. In contrast to the usual on shell renormalization scheme where the VEV is introduced as a free parameter and is fixed from the no tadpole condition, by treating the $O(\hbar^n)$ ($n > 1$) quantity which appears in the broken symmetry phase of the Lagrangian due to the VEV as a finite counter-terms, the tadpole terms vanish automatically[6] in our approach. In Sec.II, we first prove the RG invariance of the pole mass starting from the RG equation in the MS scheme and obtain the pole mass in the symmetric and broken symmetry phase. The resulting pole mass in the symmetric and broken symmetry phase satisfies the RG equation with the RG functions of the symmetric phase as shown recently in case of the EP[7]. In Sec.III, we give some discussions and conclusions.

II. RG INVARIANCE OF THE POLE MASS

In this section, we will prove the RG invariance of the pole mass of the neutral scalar field theory and demonstrate it explicitly by calculating the pole mass both in the symmetric and in broken symmetry phase. The generalization to the other cases will be straightforward. The renormalized effective action $\Gamma_R^{MS}[\phi]$ of the neutral scalar field theory in the minimal subtraction scheme(MS) is independent of the renormalization mass scale μ and satisfies the RG equation[8]

$$(D + \gamma^{MS} \phi_z \frac{\delta}{\delta \phi_z}) \Gamma_R^{MS}(\mu, \lambda, m^2, \phi) = 0, \quad (1)$$

where

$$D \equiv \mu \frac{\partial}{\partial \mu} + \beta_\lambda^{MS} \frac{\partial}{\partial \lambda} + \beta_{m^2}^{MS} \frac{\partial}{\partial m^2}, \quad (2)$$

and we use the notation that the repeated letters mean the integration over continuous variables. Here β_λ^{MS} , $\beta_{m^2}^{MS}$ and γ^{MS} are the RG functions in the MS scheme. In the broken symmetry phase ($m^2 < 0$), the scalar field ϕ develop a non-vanishing vacuum expectation value (VEV) v satisfying

$$\left[\frac{\delta \Gamma_R^{MS}(\mu, \lambda, m^2, \phi)}{\delta \phi} \right]_{\phi=v} = 0, \quad (3)$$

from which one can determine v as a function of μ, λ and m^2 . Then by shifting $\phi \rightarrow \phi + v$, we obtain the effective action in the broken symmetry phase $\Gamma_R^{SB}(\mu, \lambda, m^2, \phi)$ as

$$\Gamma_R^{SB}(\mu, \lambda, m^2, \phi) \equiv \Gamma_R^{MS}(\mu, \lambda, m^2, \phi + v(\mu, \lambda, m^2)), \quad (4)$$

which satisfy the RG equation with the RG functions as in the symmetric phase[7] so that

$$(D + \gamma^{MS} \phi \frac{\partial}{\partial \phi}) \Gamma_R^{SB}(\mu, \lambda, m^2, \phi) = 0. \quad (5)$$

The renormalized N-point one-particle-irreducible (1PI) vertex can be obtained from the effective action $\Gamma_R^{(N)}(\mu, \lambda, m^2, \phi)$ by taking the functional derivative with respect to ϕ N-times and putting $\phi = 0$ so that

$$\Gamma_R^{(N)}(\mu, \lambda, m^2) = \left[\frac{\delta^N \Gamma_R(\mu, \lambda, m^2, \phi)}{\delta \phi_{x1} \delta \phi_{x2} \cdots \delta \phi_{xN}} \right]_{\phi=0}. \quad (6)$$

By using Eqs.(1) and (6), one can obtain the RG equation satisfied by $\Gamma_R^{(N)}(\mu, \lambda, m^2)$ as

$$(D + N\gamma^{MS}) \Gamma_R^{(N)}(\mu, \lambda, m^2) = 0. \quad (7)$$

Now, the pole mass M is defined as the pole of the Green function G which is the inverse of $\Gamma_R^{(2)}$ so that

$$[G^{-1}]_{p^2=-M^2} = [p^2 + \overline{m}^2 + \Pi_R(p^2, \mu, \lambda, \overline{m}^2)]_{p^2=-M^2} = -M^2 + \overline{m}^2 + \Pi_R(-M^2, \mu, \lambda, \overline{m}^2) = 0, \quad (8)$$

where \overline{m}^2 is the mass term of the tree level Lagrangian which becomes m^2 in the symmetric phase ($m^2 > 0$) and $-2m^2$ in the broken symmetry phase ($m^2 < 0$) and $\Pi_R(p^2, \mu, \lambda, m^2)$ is the renormalized self energy obtained from the 1PI two-point Feynman diagrams. In perturbative calculations, we substitute the expansion of $\Pi_R(p^2, \mu, \lambda, \overline{m}^2)$ and M^2 in \hbar

$$\begin{aligned} \Pi_R(p^2, \mu, \lambda, \overline{m}^2) &= \hbar \Pi_{R1}(p^2, \mu, \lambda, \overline{m}^2) + \hbar^2 \Pi_{R2}(p^2, \mu, \lambda, \overline{m}^2) + \cdots, \\ M^2 &= M_0^2 + \hbar M_1^2 + \hbar^2 M_2^2 + \cdots, \end{aligned} \quad (9)$$

into Eq.(8) and obtain M^2 as a function of μ, λ and \overline{m}^2 as

$$M^2(\mu, \lambda, \overline{m}^2) = \overline{m}^2 + \hbar [\Pi_{R1}(p^2, \mu, \lambda, \overline{m}^2)]_{p^2=-\overline{m}^2} + \hbar^2 \left[\Pi_{R2}(p^2, \mu, \lambda, \overline{m}^2) - \Pi_{R1}(p^2, \mu, \lambda, \overline{m}^2) \frac{\partial \Pi_{R1}(p^2, \mu, \lambda, \overline{m}^2)}{\partial p^2} \right]_{p^2=-\overline{m}^2} + O(\hbar^3). \quad (10)$$

Now, the RG equation for G^{-1} can be obtained by putting $N=2$ in Eq.(7) as

$$0 = (D + 2\gamma^{MS})(p^2 + \overline{m}^2 + \Pi_R(p^2, \mu, \lambda, \overline{m}^2)) = D(\overline{m}^2 + \Pi_R(p^2, \mu, \lambda, \overline{m}^2)) + 2\gamma^{MS}(p^2 + \overline{m}^2 + \Pi_R(p^2, \mu, \lambda, \overline{m}^2)). \quad (11)$$

By applying the operation D to the second equation of Eq.(8), we obtain

$$0 = D(-M^2 + \overline{m}^2 + \Pi_R(-M^2, \mu, \lambda, \overline{m}^2)) = (DM^2)(-1 + \frac{\partial \Pi_R(-M^2, \mu, \lambda, \overline{m}^2)}{\partial M^2}) + [D(p^2 + \overline{m}^2 + \Pi_R(p^2, \mu, \lambda, \overline{m}^2))]_{p^2=-M^2}. \quad (12)$$

By substituting Eq.(11) into (12), we obtain

$$0 = (DM^2)(-1 + \frac{\partial \Pi_R(-M^2, \mu, \lambda, \overline{m}^2)}{\partial M^2}) - [2\gamma^{MS}(p^2 + \overline{m}^2 + \Pi_R(p^2, \mu, \lambda, \overline{m}^2))]_{p^2=-M^2}. \quad (13)$$

The last term of the of the above equation vanishes due to the definition of the pole mass (Eq.(8)) and as a result, we obtain

$$DM^2 = 0, \quad (14)$$

which means that the pole mass is RG invariant.

Now let us consider the case of the neutral scalar field theory with the Euclidean classical action

$$S[\phi] = \int d^4x (\frac{1}{2} Z_\phi \phi (-\partial^2 + m_B^2) \phi + \frac{1}{24} \lambda_B Z_\phi^2 \phi^4). \quad (15)$$

Up to two-loop, the renormalized self energy $\Pi_R(p^2, \mu, \lambda, m^2)$ of the neutral scalar field theory is given by

$$\Pi_R(p^2, \mu, \lambda, m^2) = \text{---}\times\text{---} + \frac{1}{2} \text{---}\bigcirc\text{---} + \frac{1}{2} \text{---}\bigcirc\text{---} + \frac{1}{2} \text{---}\bigcirc^{\times}\text{---} + \frac{1}{8} \text{---}\bigcirc\text{---} - \frac{1}{12} \text{---}\bigcirc\text{---}, \quad (16)$$

where the cross in the line means the counter-term for the propagator $(Z_\phi - 1)p^2 + Z_\phi m_B^2 - m^2$ and the box means the vertex counter-term $\lambda_B Z_\phi^2 - \lambda$. By using the well known MS counter-terms given by

$$\begin{aligned}\lambda_B &= \mu^{2\varepsilon} \left\{ \lambda + \frac{\hbar}{(4\pi)^2} \frac{3\lambda^2}{2\varepsilon} + \frac{\hbar^2}{(4\pi)^4} \left(-\frac{\lambda^2}{2\varepsilon^2} + \frac{\lambda^2}{2\varepsilon^2} \right) \cdots \right\}, \\ m_B^2 &= m^2 \left\{ 1 + \frac{\hbar}{(4\pi)^2} \frac{\lambda}{2\varepsilon} + \frac{\hbar^2}{(4\pi)^4} \left(-\frac{5\lambda^2}{24\varepsilon} + \frac{\lambda^2}{2\varepsilon^2} \right) + \cdots \right\}, \\ Z_\phi &= \mu^{-\varepsilon} \left\{ 1 - \frac{\hbar^2}{(4\pi)^4} \frac{\lambda^2}{24\varepsilon^2} \cdots \right\},\end{aligned}\tag{17}$$

we can obtain

$$\begin{aligned}\Pi_{R1}(p^2, \mu, \lambda, m^2) &= \frac{\lambda m^2}{32\pi^2} \left\{ \ln\left(\frac{m^2}{\mu^2}\right) - 1 \right\}, \\ \Pi_{R2}(p^2, \mu, \lambda, m^2) &= \frac{\lambda^2 m^2}{(16\pi^2)^2} \left\{ \frac{1}{2} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{5}{4} \ln\left(\frac{m^2}{\mu^2}\right) + \frac{1}{12} p^2 \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{6} J_3\left(\frac{m^2}{p^2}\right) \right\},\end{aligned}\tag{18}$$

where $J_3(\frac{m^2}{p^2})$ can be found in [9]. By using (10) one can obtain the expansion of the pole mass M^2 up to the order \hbar^2 as

$$M^2 = m^2 + \frac{\hbar}{32\pi^2} \lambda m^2 \left\{ \ln\left(\frac{m^2}{\mu^2}\right) - 1 \right\} + \frac{\hbar^2 \lambda^2}{(16\pi^2)^2} \lambda^2 m^2 \left\{ \frac{1}{2} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{7}{6} \ln\left(\frac{m^2}{\mu^2}\right) - \frac{1}{6} J_3(1) \right\} \cdots.\tag{19}$$

By using the RG functions in MS scheme [8]

$$\beta_\lambda^{MS} = \mu \frac{d\lambda}{d\mu} = 3 \frac{\hbar}{(4\pi)^2} \lambda^2 - \frac{17}{3} \frac{\hbar^2}{(4\pi)^4} \lambda^3 + \left(\frac{145}{8} + 12 \zeta(3) \right) \frac{\hbar^3}{(4\pi)^6} \lambda^4 + \cdots,\tag{20}$$

$$\beta_{m^2}^{MS} = \frac{\mu}{m^2} \frac{dm^2}{d\mu} = \frac{\hbar}{(4\pi)^2} \lambda - \frac{5}{6} \frac{\hbar^2}{(4\pi)^4} \lambda^2 + \frac{7}{2} \frac{\hbar^3}{(4\pi)^6} \lambda^3 + \cdots,\tag{21}$$

$$\gamma^{MS} = \frac{\mu}{\phi} \frac{d\phi}{d\mu} = -\frac{1}{12} \frac{\hbar^2}{(4\pi)^4} \lambda^2 + \frac{1}{16} \frac{\hbar^3}{(4\pi)^6} \lambda^3 + \cdots,\tag{22}$$

we can confirm the RG invariance of the pole mass M^2 given in Eq.(19).

Now, let us consider the case of broken symmetry phase ($m^2 < 0$). The Euclidean action for the broken symmetry phase can be obtained from that of the symmetric phase by shifting the field $\phi \rightarrow \phi + v$. In case of the neutral scalar field theory, the perturbative expansion of v can be obtained as [7]

$$v(\mu, \lambda, m^2) = v_0(\lambda, m^2) + \hbar v_1(\mu, \lambda, m^2) + \cdots,\tag{23}$$

where

$$v_0^2 = -m^2 \frac{6}{\lambda} \text{ and } v_0 v_1 = \frac{3}{16\pi^2} \lambda m^2 \left\{ \ln\left(\frac{-2m^2}{\mu^2}\right) - 1 \right\}.\tag{24}$$

As a result, the $O(\hbar)$ terms of the classical action contains both the usual $\frac{1}{\varepsilon}$ divergent counter-terms and the finite counter-terms coming from the $O(\hbar)$ terms of the VEV's so that at one-loop we obtain

$$\begin{aligned}S[\phi] &= \int d^4x \left[\frac{1}{2} \phi (-\partial^2 + m^2 + \frac{1}{2} \lambda v_0^2) \phi + \frac{1}{6} \lambda v_0 \phi^3 + \frac{1}{24} \lambda \phi^4 + (Z_\phi m_B^2 v + \frac{1}{6} \lambda_B Z_\phi^2 v^3) \phi \right. \\ &\quad \left. + ((Z_\phi - 1)p^2 + Z_\phi m_B^2 - m^2 + \frac{1}{2} (\lambda_B Z_\phi^2 v^2 - \lambda v_0^2)) \phi^2 + \frac{1}{6} (\lambda_B Z_\phi^2 v - \frac{\lambda}{6} v_0) \phi^3 + (\lambda_B Z_\phi^2 - \lambda) \phi^4 \right] \\ &= \int d^4x \left[\frac{1}{2} \phi (-\partial^2 - 2m^2) \phi + \frac{1}{6} \lambda v_0 \phi^3 + \frac{1}{24} \lambda \phi^4 + \hbar \left\{ \left(-\frac{1}{(4\pi)^2} \frac{\lambda}{\varepsilon} v_0 - 2v_1 \right) m^2 \phi \right. \right. \\ &\quad \left. \left. + \left(-\frac{1}{(4\pi)^2} \frac{2\lambda}{\varepsilon} m^2 + \frac{1}{2} \lambda v_0 v_1 \right) \phi^2 + \left(\frac{1}{(4\pi)^2} \frac{\lambda^2}{4\varepsilon} v_0 + \frac{1}{6} \lambda v_1 \right) \phi^3 + \frac{1}{(4\pi)^2} \frac{\lambda^2}{16\varepsilon} \phi^4 \right\} + O(\hbar^2) \right],\end{aligned}\tag{25}$$

where we have used Eq.(17) to obtain the one-loop counter-terms given in the last line. The one-loop two-point function is given by

$$\Pi_{R1}(p^2, \mu, \lambda, m^2) = \text{---}\text{X}\text{---} + \frac{1}{2} \text{---}\bigcirc\text{---} - \frac{1}{2} \text{---}\bigcirc\text{---} + \frac{1}{2} \text{---}\bigcirc\text{---} + \text{---}\text{---}\text{---}, \quad (26)$$

where the four-point vertex has the vertex factor λ , the three-point vertex has the vertex factor λv_0 , the filled circle means the tadpole counter-term $Z_\phi m_B^2 v + \frac{1}{6} \lambda_B Z_\phi^2 v^3$ and the cross in the line means the counter-term for the propagator $(Z_\phi - 1)p^2 + Z_\phi m_B^2 - m^2 + \frac{1}{2}(\lambda_B Z_\phi^2 v^2 - \lambda v_0^2)$. As noted in introduction, the last two terms of the above equation which are the one-loop tadpole terms cancel out exactly and the sum of the first three terms of the above equation give finite result as

$$\Pi_{R1}(p^2, \mu, \lambda, m^2) = \frac{2\lambda m^2}{16\pi^2} (\ln(\frac{-2m^2}{\mu^2}) + 1) - \frac{3\lambda m^2}{16\pi^2} \int_0^1 d\alpha \ln\{\frac{-2m^2 + \alpha(1-\alpha)p^2}{\mu^2}\}. \quad (27)$$

Note that the $\frac{1}{\epsilon}$ divergences of the two-point Green function in the broken symmetry phase cancel out by the counter-terms of the symmetric phase given in Eq.(17)[10] and that the two-point Green function in the broken symmetry phase satisfies the RG equation with the RG functions of the symmetric phase given in Eq.(14). The pole mass in the broken symmetry phase can be obtained from Eq.(10) as

$$M^2 = -2m^2 + \frac{\hbar}{16\pi^2} \lambda m^2 \{-\ln(\frac{-2m^2}{\mu^2}) + 8 - \sqrt{3}\pi\} + O(\hbar^2), \quad (28)$$

which satisfy the RG equation given in Eq.(14) with the RG functions of the symmetric phase.

From the RG invariance of the pole mass, we can use the method of characteristics[11] to obtain the RG improvement of the pole mass in the perturbative calculations as

$$M^2(\mu, \lambda, m^2) = M^2(\mu e^{2t}, \lambda(t), m^2(t)), \quad (29)$$

where $\lambda(t)$ and $m^2(t)$ satisfies

$$\begin{aligned} \frac{d\lambda(t)}{dt} &= \beta_\lambda^{MS}(\lambda(t)), \\ \frac{dm^2(t)}{dt} &= \beta_{m^2}^{MS}(\lambda(t)), \end{aligned} \quad (30)$$

with the initial conditions $\lambda(0) = \lambda$ and $m^2(0) = m$.

III. DISCUSSIONS AND CONCLUSIONS

In this paper, we have proved the RG invariance of the pole mass and have verified it explicitly from perturbative calculations in case of the scalar field theory both in the symmetric and in broken symmetry phase. In case of the broken symmetry phase, we have used an approach where the VEV was obtained from the EP in the symmetric phase as a function of the Lagrangian parameters and then the Lagrangian in broken symmetry phase was obtained by shifting the scalar field in contrast to the usual on shell renormalization scheme where the VEV introduced as a free parameter in the Lagrangian of the broken symmetry phase, is fixed by the no tadpole condition. As a result, we have shown that the resulting pole mass both in the symmetric and in broken symmetry phase satisfies the RG equation with RG function of the symmetric phase. The RG invariance of the pole mass in the broken symmetry phase by using the RG functions of the symmetric phase can be used as a check to the higher order calculation of the pole mass as well as RG improvement of the pole mass.

Acknowledgments

This research was supported in part by the Institute of Natural Science.

[1] Narison, see M. Sher, Phys. Rep. 179, 273 (1989).

- [2] A. Sirlin and R. Zucchini, Nucl. Phys. B266 (1986) 389.
- [3] F. Jegerlehner, M. Yu. Kalmykov and O. L. Veretin, Nucl. Phys. B641 (2002) 285; B658 (2003) 49.
- [4] A. S. Kronfeld, Phys. Rev. D58 (1998) 051501.
- [5] P. Kielanowski and S. R. Juarez W, Phys. Rev. D72 (2005) 096003.
- [6] J. C. Taylor, *Gauge Theories of Weak Interactions* (Cambridge Press, Cambridge, 1976).
- [7] C. Kim, Joul. of the Kor. Phys. Soc. 59 (2011) 2993.
- [8] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin, S. A. Larin, Phys. Lett. B**272**, 39(1991); ibid. 319, 545(1993).
- [9] J. Fleischer, F. Jegerlehner, O. V. Tarasov and O. L. Veretin , Nucl. Phys. B539 (1999) 671
- [10] K. Symanzik, Proc. of "Fundamental Interactions at High Energies" ed. A. Perlmutter (Gordon and Breach, New York, 1970).
- [11] C. Ford, D. R.T. Jones, P. W. Stephenson and M. B. Einhorn, Nucl. Phys. B395 (1993) 17.